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Statistical Equilibrium States and Long-Time Dynamics for a Transport Equation

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Statistical equilibrium states for a linear transport equation were defined in a previous work. We consider here the two-dimensional case: we show that under some mild assumptions these equilibrium states actually describe the long-time dynamics of the system.

KEY WORDS: Statistical equilibrium states; transport equation; long-time dynamics.

1. INTRODUCTION

We shall consider in this paper the linear transport equation

$$(\mathscr{T}) \quad \begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 & \text{on } \Omega \\ \rho(0, \mathbf{x}) = \rho_0(\mathbf{x}) \end{cases}$$

where Ω is a bounded, connected, and regular open domain of $\mathbb{R}^2$ [we shall assume $d\mathbf{x}(\Omega) = 1$ for simplicity]; $\mathbf{u}(\mathbf{x})$ is a given incompressible ($\nabla \cdot \mathbf{u} = 0$) velocity field in $C^1(\bar{\Omega})$ which satisfies $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$ ($\mathbf{n}$ is the normal unit vector at the boundary $\partial\Omega$); and $\rho(t, \mathbf{x})$ is a scalar function.

It is well known that for any given $\rho_0(\mathbf{x})$ in $L^\infty(\Omega)$, the equation $(\mathscr{T})$ has a unique weak solution $\rho(t, \mathbf{x})$ given by $\rho(t, \mathbf{x}) = \rho_0(\varphi_t^{-1}(\mathbf{x}))$, where φ_t is the Lagrangian flow associated to $\mathbf{u}$:

$$\begin{cases} \dfrac{d}{dt} \varphi_t(\mathbf{x}) = \mathbf{u}(\varphi_t(\mathbf{x})) \\ \varphi_0(\mathbf{x}) = \mathbf{x} \end{cases}$$

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Since $\mathbf{u}$ is incompressible, $\varphi_t: \Omega \to \Omega$ conserves the Lebesgue measure $d\mathbf{x}$.

We denote by Φ_t the flow on $L^\infty(\Omega)$ defined by $(\mathscr{T})$:

$$\rho(t, \mathbf{x}) = (\Phi_t \rho_0)(\mathbf{x})$$

We will investigate here the long-time behavior of this flow. It is easily observed that besides some degenerate cases (mainly the case where $\mathbf{u}$ is the velocity field of a solid-body rotation), the solution of $(\mathscr{T})$ undergoes small-scale oscillations and converges (in a weak sense), when t goes to infinity, toward some final state. We will give here a mathematical proof of the convergence of the flow, and show that the final state is accurately described as a statistical equilibrium state, according to the theory given in ref. 2.

Although this problem is interesting in itself, it is enlightening to consider it in the context of 2D turbulence: equation $(\mathscr{T})$ describes the particular case of a passive scalar convected by a frozen velocity field. Our result points out a simple example where the "topological invariants" (such as the number of patches or holes) are clearly irrelevant to the long-time behavior. And we may conjecture that these invariants have no influence on the final state in the case of the 2D Euler system.

2. STATISTICAL EQUILIBRIUM STATES FOR $(\mathscr{T})$

As previously noticed,[2] $(\mathscr{T})$ belongs to a class of equations to which a statistical equilibrium theory can be applied. Therefore to any given $\rho_0(\mathbf{x})$ in $L^\infty(\Omega)$ we can associate a statistical equilibrium state which is a Young measure ν^* (see ref. 2 and below). We will prove in the following section that, under some mild assumptions on the field $\mathbf{u}$, the Young measure ν^* describes also the long-time dynamics of the system. Thus, that, in this particular case, a precise link between the long-time dynamics and the statistical equilibrium theory is established.

We refer to ref. 2 for a detailed presentation of the theory; we shall only indicate here the main recipe to get the equilibrium states.

2.1. Constants of the Motion for $(\mathscr{T})$

Let us denote by ψ the stream function of $\mathbf{u}$ [$\mathbf{u} = \nabla \times (\psi \mathbf{k})$, $\psi = 0$ on $\partial\Omega$, with $\mathbf{k}$ the unit vector normal to the plane].

For any bounded continuous function $g(\psi, \lambda)$ on $\mathbb{R} \times \mathbb{R}$, we define the functional

$$F_g(\rho) = \int_\Omega g(\psi(\mathbf{x}), \rho(\mathbf{x}))\, d\mathbf{x}$$

One easily checks that F_g is conserved by the flow Φ_t. Indeed,

$$F_g(\Phi_t \rho) = \int_\Omega g(\psi(\mathbf{x}), \rho(\varphi_t^{-1}(\mathbf{x})))\, d\mathbf{x}$$

We make the change of variable $\mathbf{x} = \varphi_t(\mathbf{x}')$; since ψ is the stream function of $\mathbf{u}$, we have $\psi(\varphi_t(\mathbf{x}')) = \psi(\mathbf{x}')$, so that $F_g(\Phi_t \rho) = F_g(\rho)$.

2.2. The Variational Problem

Let us recall a few definitions.[2,5]

• A Young measure ν on $\Omega \times [-R, R]$ is a measurable mapping $\mathbf{x} \to \nu_\mathbf{x}$ from Ω into the space $M_1([-R, R])$ of the Borel probability measures on $[-R, R]$, endowed with the narrow topology (weak topology associated to the continuous bounded functions). We denote by $\mathscr{M}_R$ the convex set of the Young measures on $\Omega \times [-R, R]$; $\mathscr{M}_R$ is endowed with the narrow topology (of bounded measures on $\Omega \times [-R, R]$) and it is a compact space.

• To any measurable function $\rho: \Omega \to [-R, R]$ (i.e., $\rho \in L_R^\infty(\Omega)$), we associate the Young measure $\delta_\rho: \mathbf{x} \to \delta_{\rho(\mathbf{x})}$, the Dirac mass at $\rho(\mathbf{x})$.

• For $\rho_0 \in L_R^\infty(\Omega)$, we define the probability distribution π_0 on $[-R, R]$ by

$$\langle \pi_0, f \rangle = \int_\Omega f(\rho_0(\mathbf{x}))\, d\mathbf{x}$$

and we denote by π the Young measure such that $\pi_\mathbf{x} = \pi_0$, for all $\mathbf{x}$.

• For $\nu \in \mathscr{M}_R$, the Kullback information functional $I_\pi(\nu)$ is defined by[4]

$$I_\pi(\nu) = \int_\Omega d\mathbf{x} \int \mathrm{Ln}\left(\frac{d\nu_x}{d\pi_0}\right) d\nu_\mathbf{x}$$

if ν is absolutely continuous with respect to π

$$I_\pi(\nu) = +\infty \qquad \text{if not}$$

Let us suppose now that $\delta_{\Phi_t \rho_0} \to \nu$ (when $t \to +\infty$) in the space $\mathscr{M}_R$, so that

$$\int_\Omega f(\mathbf{x}, \Phi_t \rho_0(\mathbf{x}))\, d\mathbf{x} \to \int_\Omega d\mathbf{x} \int f(\mathbf{x}, \lambda)\, d\nu_\mathbf{x}(\lambda)$$

for any continuous bounded function $f(\mathbf{x}, \lambda)$. In particular, if we take $f(\mathbf{x}, \lambda) = g(\psi(\mathbf{x}), \lambda)$, we get

$$\int_\Omega d\mathbf{x} \int g(\psi(\mathbf{x}), \lambda)\, d\nu_{\mathbf{x}}(\lambda) = \int_\Omega g(\psi(\mathbf{x}), \rho_0(\mathbf{x}))\, d\mathbf{x} \tag{1}$$

for any continuous bounded function $g(\psi, \lambda)$.

We denote by $\mathscr{E}$ the subset of $\mathscr{M}_R$ composed of the Young measures satisfying (1). $\mathscr{E}$ is convex, compact, and nonempty.

Now, following the approach given in ref. 2, the equilibrium state ν^* is defined as the solution of the variational problem

$$\text{(V.P.)} \qquad I_\pi(\nu^*) = \min_{\nu \in \mathscr{E}} I_\pi(\nu)$$

Since I_π is a (positive) convex lower-semicontinuous functional on $\mathscr{M}_R$ which is strictly convex on $\{\nu\colon I_\pi(\nu) < +\infty\}$, (V.P.) has always a solution, which is unique if I_π is not identical to $+\infty$ on $\mathscr{E}$.

We describe now the solution of this variational problem.

2.3. The Young Measure π^Ψ

The functional $g(\psi, \lambda) \to \int_\Omega g(\psi(\mathbf{x}), \rho_0(\mathbf{x}))\, d\mathbf{x}$ defines a Borel probability measure on $\mathbb{R} \times \mathbb{R}$, whose first projection is the image measure P of $d\mathbf{x}$ by ψ, i.e.,

$$\int_\Omega g(\psi(\mathbf{x}))\, d\mathbf{x} = \int g(\psi)\, dP(\psi)$$

Therefore, by a general theorem,[1] there is a unique Young measure $\psi \to \zeta_\psi$ on $\mathbb{R} \times \mathbb{R}$, such that

$$\int_\Omega g(\psi(\mathbf{x}), \rho_0(\mathbf{x}))\, d\mathbf{x} = \int dP(\psi) \int g(\psi, \lambda)\, d\zeta_\psi(\lambda)$$

The Young measure π^ψ on $\Omega \times [-R, R]$ is now defined by $\pi^\psi_{\mathbf{x}} = \zeta_{\psi(\mathbf{x})}$ a.e. (almost everywhere) in Ω.

Obviously π^ψ belongs to $\mathscr{E}$, and the solution of (V.P.) straightforwardly follows from the following lemma, whose proof is easy.

Lemma. Let ν be any Young measure in $\mathscr{E}$; then we have $I_\pi(\nu) = I_{\pi^\psi}(\nu) + I_\pi(\pi^\psi)$.

Therefore, if $I_\pi(\pi^\psi) < +\infty$, π^ψ is the unique solution of (V.P.).

Remark. We can easily check that the condition $I_\pi(\pi^\psi) < +\infty$ is satisfied if ρ_0 takes a finite number of values, but we do not know whether it is always satisfied. If $I_\pi(\pi^\psi) = +\infty$, π^ψ is still a solution of (V.P.), but it is no longer unique.

3. LONG-TIME DYNAMICS

It is a general underlying assumption in statistical mechanics that the equilibrium state describes, in some sense, the long-time dynamics of the system. More precisely, we expect here that $\delta_{\Phi_t \rho_0} \to \pi^\psi$ (when $t \to \infty$) in the space $\mathscr{M}_R$.

We prove now that this is true, under some mild assumptions on the velocity field $\mathbf{u}$ (which mainly consists in discarding the degenerate case of a solid-body rotation).

3.1. Preliminary Assumptions on *u*

- The stream function ψ varies from 0 (value taken at the boundary $\partial\Omega$) to its maximum value $\psi_m > 0$, which is reached at a unique point $\mathbf{x}^* \in \Omega$.
- $\nabla\psi(\mathbf{x}) \neq 0$, for all $\mathbf{x} \neq \mathbf{x}^*$, $\mathbf{x} \in \bar{\Omega}$.
- For all ψ, $0 \leqslant \psi < \psi_m$, the set $\Sigma_\psi = \{\mathbf{x} \in \bar{\Omega}\colon \psi(\mathbf{x}) = \psi\}$ is a closed regular curve. We denote by $d\sigma$ the superficial measure on Σ_ψ, and by $d\sigma_\psi = d\sigma/|\nabla\psi|$ the φ_t-invariant measure.
- We denote by $T_\psi = \int_{\Sigma_\psi} d\sigma_\psi$ the time needed to cover Σ_ψ. We assume that T_ψ is a regular function of ψ on the interval $[0, \psi_m[$; it is integrable since

$$\int_0^{\psi_m} T_\psi \, d\psi = \int_0^{\psi_m} d\psi \int_{\Sigma_\psi} d\sigma_\psi = \int_\Omega d\mathbf{x} = 1$$

We can now state our result.

Theorem. Let us assume that, besides the above assumptions, the following hypothesis (H) holds:

$$\text{(H)} \qquad \left(\int_\psi^{\psi_m} T_\psi \, d\psi\right)^{1/2} \left|\frac{d}{d\psi}\left(\frac{1}{T_\psi^2}\right)\right| \geqslant c > 0 \qquad \text{for} \quad 0 \leqslant \psi < \psi_m$$

Then, for any $\rho_0 \in L_R^\infty(\Omega)$, we have

$$\delta_{\Phi_t \rho_0} \xrightarrow[(t \to \infty)]{} \pi^\psi \qquad \text{in the space } \mathscr{M}_R$$

Proof. Let us first clear up the meaning of the hypothesis (H), which is *a priori* rather obscure. With this aim, let us take $\Omega = D(0, \pi^{-1/2})$, disk of radius $\pi^{-1/2}$ centered at 0, and $\psi = \psi(r)$, a radially symmetric function. Let us denote by $\alpha(r) = -\psi'(r)/r$ the angular velocity of the motion on the circle of radius r. Elementary calculations give

$$\alpha'(r) = -2\pi^{3/2}\left(\int_{\psi(r)}^{\psi_m} T_\psi \, d\psi\right)^{1/2} \frac{d}{d\psi}\left(\frac{1}{T_\psi^2}\right)$$

so that (H) is written $|\alpha'(r)| \geqslant \alpha_0 > 0$ for $0 < r \leqslant \pi^{-1/2}$.

The proof of the theorem is in two steps. First we prove the result for a disk and a radially symmetric motion (proposition below), next we deduce the general case by an appropriate transformation.

Proposition. Let $\Omega = D(0, \pi^{-1/2})$ and ψ be radially symmetric. Let us suppose that $|\alpha'(r)| \geqslant \alpha_0 > 0$ for $0 < r \leqslant \pi^{-1/2}$. Then for any $\rho_0 \in L_R^\infty(\Omega)$, we have

$$\delta_{\Phi_t \rho_0} \xrightarrow[(t \to \infty)]{} \pi^\psi \qquad \text{in the space } \mathscr{M}_R$$

Proof. Let us first remark that it suffices to show that

$$\langle \delta_{\Phi_t \rho_0}, f \rangle \to \langle \pi^\psi, f \rangle$$

for any smooth, compactly supported $f(\mathbf{x}, \lambda)$ on $\Omega \times \mathbb{R}$.

We may now prove the result for a continuous ρ_0 with compact support in Ω: indeed, let $\rho_0 \in L^\infty(\Omega)$ and f as above; then for any $\varepsilon > 0$ there exists ρ_0^ε continuous with compact support such that

$$\|\rho_0 - \rho_0^\varepsilon\|_{L^1(\Omega)} \leqslant \varepsilon$$

and so

$$|\langle \delta_{\Phi_t \rho_0}, f \rangle - \langle \delta_{\Phi_t \rho_0^\varepsilon}, f \rangle| \leqslant K_f \varepsilon$$

where K_f is the Lipschitz constant of f in the second variable, as well as

$$|\langle \pi^\psi, f \rangle - \langle \pi_\varepsilon^\psi, f \rangle| \leqslant K_f \varepsilon$$

with π_ε^ψ associated to ρ_0^ε.

We thus obtain

$$|\langle \delta_{\Phi_t \rho_0}, f \rangle - \langle \pi^\psi, f \rangle| \leqslant 2K_f \varepsilon + |\langle \delta_{\Phi_t \rho_0^\varepsilon}, f \rangle - \langle \pi_\varepsilon^\psi, f \rangle|$$

From now on we suppose ρ_0 continuous with compact support.

Let $\mathscr{A}^{m,n}$ be the following partition of $\Omega\backslash\{0\}$:

$$\mathscr{A}^{m,n}=\{A_{k,j}^{m,n},\ 0\leqslant k<m,\ 0\leqslant j<n\}$$

where $A_{k,j}^{m,n}$ is the set of those points whose polar coordinates (r, θ) satisfy

$$\begin{cases} \dfrac{2k\pi}{m}\leqslant\theta<\dfrac{2(k+1)\pi}{m} \\ \dfrac{j}{n\sqrt{\pi}}\leqslant r<\dfrac{j+1}{n\sqrt{\pi}} \end{cases}$$

(the first inequality on the radius being strict for $j=0$).

Define $\rho_0^{m,n}$ on $\Omega\backslash\{0\}$ by

$$\rho_0^{m,n}(\mathbf{x})=\frac{1}{|A_{k,j}^{m,n}|}\int_{A_{k,j}^{m,n}}\rho(\mathbf{x}')\,d\mathbf{x}' \qquad \text{if} \quad \mathbf{x}\in A_{k,j}^{m,n}$$

and $f^{m,n}$ in the same way.

It is obvious that $\|\rho_0-\rho_0^{m,n}\|_\infty$ and $\|f-f^{m,n}\|_\infty$ both converge to 0 as m and n tend to infinity.

One easily sees that

$$\begin{aligned}&|\langle\delta_{\Phi_t\rho_0}, f\rangle-\langle\pi^\psi, f\rangle|\\ &\quad\leqslant|\langle\delta_{\Phi_t\rho_0^{m,n}}, f^{m,n}\rangle-\langle\pi^\psi_{m,n}, f^{m,n}\rangle|+2\,\|f-f^{m,n}\|_\infty+2K_f\,\|\rho_0-\rho_0^{m,n}\|_\infty\end{aligned}$$

For fixed m, n we shall give an upper bound for

$$\operatorname*{Lim\,sup}_{t\to\infty}|\langle\delta_{\Phi_t\rho_0^{m,n}}, f^{m,n}\rangle-\langle\pi^\psi_{m,n}, f^{m,n}\rangle|$$

One has

$$\begin{aligned}\langle\delta_{\Phi_t\rho_0^{m,n}}, f^{m,n}\rangle&=\sum_{k,j}\int_{A_{k,j}^{m,n}}f^{m,n}(\mathbf{x}_{k,j}, \rho_0^{m,n}(\varphi_t^{-1}(\mathbf{x}))\,d\mathbf{x}\\ &=\sum_{i,k,j}\int_{A_{i,j}^{m,n}\cap\varphi_t^{-1}(A_{k,j}^{m,n})}f^{m,n}(\mathbf{x}_{k,j}, \rho_0^{m,n}(\mathbf{x}_{i,j})\,d\mathbf{x}\end{aligned}$$

where $\mathbf{x}_{k,j}$ is any fixed point in $A_{k,j}^{m,n}$, so

$$\langle\delta_{\Phi_t\rho_0^{m,n}}, f^{m,n}\rangle=\sum_{i,k,j}|A_{i,j}^{m,n}\cap\varphi_t^{-1}(A_{k,j}^{m,n})|\,f^{m,n}(\mathbf{x}_{k,j}, \rho_0^{m,n}(\mathbf{x}_{i,j}))$$

On the other hand,

$$\langle \pi^{\psi}_{m,n}, f^{m,n} \rangle = \int_{\Omega} d\mathbf{x} \frac{1}{T_{\psi(\mathbf{x})}} \int_{\Sigma_{\psi(\mathbf{x})}} f^{m,n}(\mathbf{x}, \rho_0^{m,n}(\mathbf{x}'))\, d\sigma_{\psi}(\mathbf{x}')$$
$$= \sum_{i,k,j} \frac{|A^{m,n}_{k,j}|}{m} f^{m,n}(\mathbf{x}_{k,j}, \rho_0^{m,n}(\mathbf{x}_{i,j}))$$

where $|A| = d\mathbf{x}(A)$.

Hence

$$|\langle \delta_{\Phi_t \rho_0^{m,n}}, f^{m,n} \rangle - \langle \pi^{\psi}_{m,n}, f^{m,n} \rangle|$$
$$\leqslant \|f\|_{\infty} \sum_{i,k,j} \left| |A^{m,n}_{i,j} \cap \varphi_t^{-1}(A^{m,n}_{k,j})| - \frac{|A^{m,n}_{k,j}|}{m} \right|$$

We shall now use the following lemma, whose proof will be given later.

Lemma. Under the same hypotheses as in the proposition, one has

$$\operatorname*{Lim\,sup}_{t \to \infty} \left| |A^{m,n}_{i,j} \cap \varphi_t^{-1}(A^{m,n}_{k,j})| - \frac{|A^{m,n}_{k,j}|}{m} \right| \leqslant \frac{C}{n^2 m^2}$$

We deduce from this lemma that

$$\operatorname*{Lim\,sup}_{t \to \infty} |\langle \delta_{\Phi_t \rho_0^{m,n}}, f^{m,n} \rangle - \langle \pi^{\psi}_{m,n}, f^{m,n} \rangle| \leqslant \|f\|_{\infty} \sum_{i,k,j} \frac{C}{n^2 m^2} = \frac{C}{n} \|f\|_{\infty}$$

and that $\forall m, n > 0$,

$$\operatorname*{Lim\,sup}_{t \to \infty} |\langle \delta_{\Phi_t \rho_0}, f \rangle - \langle \pi^{\psi}, f \rangle|$$
$$\leqslant \frac{C}{n} \|f\|_{\infty} + 2\, \|f - f^{m,n}\|_{\infty} + 2K_f \, \|\rho_0 - \rho_0^{m,n}\|_{\infty}$$

which concludes the proof of the proposition.

Proof. Since $|\alpha'(r)| \geqslant \alpha_0 > 0$, the angular velocity is either strictly increasing or strictly decreasing with the radius. It follows that $\varphi_t^{-1}(A^{m,n}_{k,j})$ will swirl around the origin as t increases.

If $\mathbf{x}$ has (r, θ) as polar coordinates, $\varphi_t^{-1}(\mathbf{x})$ will be represented by $(r, \theta - \alpha(r)\, t)$. Let us compute the area of $A^{m,n}_{i,j} \cap \varphi_t^{-1}(A^{m,n}_{k,j})$. This set is a union of several connected components, all but at most two of them being "complete." The evaluation of the area of one of those C_{r_1, r_2, r_3} (see Fig. 1) gives

$$|C_{r_1, r_2, r_3}| = t \int_{r_1}^{r_2} |\alpha(r) - \alpha(r_1)|\, r\, dr + t \int_{r_2}^{r_3} |\alpha(r_3) - \alpha(r)|\, r\, dr$$

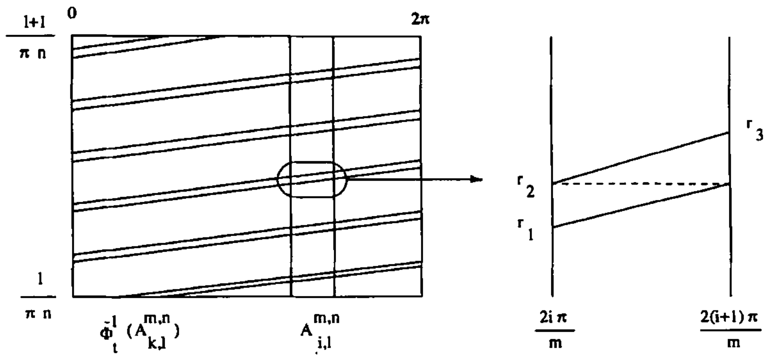

Fig. 1. Connected component C_{r_1,r_2,r_3} of $\varphi_t^{-1}(A_{k,l}^{m,n}) \cap A_{i,l}^{m,n}$.

Let us now suppose $\alpha(r)$ increasing (the decreasing case may be treated in the same way). Let us compute

$$t \int_{r_1}^{r_2} (\alpha(r) - \alpha(r_1))\, r\, dr$$

Since

$$\alpha(r) - \alpha(r_1) = (r - r_1)\left[\alpha'\left(\frac{j}{n\sqrt{\pi}}\right) + O\left(\frac{1}{n}\right)\right]$$

a straightforward calculation gives

$$t \int_{r_1}^{r_2} (\alpha(r) - \alpha(r_1))\, r\, dr = t\alpha'\left(\frac{j}{n\sqrt{\pi}}\right)\frac{j}{n\sqrt{\pi}}\frac{(r_2 - r_1)^2}{2} + O\left(t\,\frac{(r_2 - r_1)^2}{n}\right)$$

and also

$$t \int_{r_2}^{r_3} (\alpha(r_3) - \alpha(r))\, r\, dr$$
$$= t\alpha'\left(\frac{j}{n\sqrt{\pi}}\right)\frac{j}{n\sqrt{\pi}}\frac{(r_3 - r_2)^2}{2} + O\left(t\,\frac{(r_3 - r_2)^2}{n}\right)$$

On the other hand, from the relationships

$$t(\alpha(r_2) - \alpha(r_1)) = \frac{2\pi}{m}$$

and

$$t(\alpha(r_3)-\alpha(r_2))=\frac{2\pi}{m}$$

one easily gets

$$r_2-r_1=\frac{2\pi}{mt\alpha'(j/n\sqrt{\pi})}+O\left(\frac{1}{mnt}\right)$$

and the same for r_3-r_2.

So

$$|C_{r_1,r_2,r_3}|=4\pi^2\frac{j}{n\sqrt{\pi}\,m^2t\alpha'(j/n\sqrt{\pi})}+O\left(\frac{1}{m^2n^3t}\right)$$

Now the number of connected components of $A_{i,j}^{m,n}\cap\varphi_t^{-1}(A_{k,j}^{m,n})$ is equal to

$$\frac{t}{2\pi}\left[\alpha\left(\frac{j+1}{n\sqrt{\pi}}\right)-\alpha\left(\frac{j}{n\sqrt{\pi}}\right)\right]+O(1)=\frac{t}{2\pi}\frac{1}{n\sqrt{\pi}}\alpha'\left(\frac{j}{n\sqrt{\pi}}\right)+O\left(\frac{t}{n^2}\right)+O(1)$$

so that

$$|A_{i,j}^{m,n}\cap\varphi_t^{-1}(A_{k,j}^{m,n})|=\frac{2j}{m^2n^2}+O\left(\frac{1}{m^2n^5}\right)+O\left(\frac{1}{m^2n^3t}\right)$$

but $|A_{k,j}^{m,n}|=(2j+1)/mn^2$, hence

$$\left||A_{i,j}^{m,n}\cap\varphi_t^{-1}(A_{k,j}^{m,n})|-\frac{|A_{k,j}^{m,n}|}{m}\right|\leqslant\frac{C}{n^2m^2}+\frac{C}{m^2t}$$

which concludes the lemma.

Now, to prove the theorem, we construct an area-preserving homeomorphism $\Theta\colon\bar{\Omega}\to\bar{D}(0,\pi^{-1/2})$, which transforms the motion into a radially symmetric one.

We proceed as follows.

- We define the function

$$r(\psi)=\frac{1}{\sqrt{\pi}}\left(\int_{\psi}^{\psi_m}T_\psi\,d\psi\right)^{1/2}$$

• We fix an origin O_ψ on each streamline Σ_ψ such that $\psi \to O_\psi$ is a continuous curve.

• For $\mathbf{x} \in \Sigma_\psi$, $\Theta(\mathbf{x})$ is located on the circle of radius $r(\psi)$, with polar angle $(2\pi/T_\psi) \int_{O_\psi}^{\mathbf{x}} d\sigma_\psi$.

Due to our particular choice of $r(\psi)$, we easily check that Θ is an area-preserving homeomorphism; and $\Theta(\varphi_t)$ gives a uniform circular motion on each circle of radius $r(\psi)$, with the angular speed $\alpha(r(\psi)) = 2\pi/T_\psi$, from which we obtain

$$\alpha'(r(\psi)) = -2\pi^{3/2} \left(\int_\psi^{\psi_m} T_\psi \, d\psi \right)^{1/2} \frac{d}{d\psi} \left(\frac{1}{T_\psi^2} \right)$$

The theorem then straightforwardly follows from the proposition.

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